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Canonical Quantum Gravity on the Space of Vassiliev Invariants

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I review some recent results on canonical quantum gravity in the spin network representation. A set of ambient isotopy spin network invariants is introduced. These invariants are the natural extension to spin networks of the Vassiliev invariants. It is shown that this set is loop differentiable in the sense of distributions. The quantum gravity constraints are written in terms of loop derivatives. It is explicitly shown that Vassiliev invariants solve the diffeomorphism constraint. The regularized Ashtekar Hamiltonian constraint is studied, and its action on valence-four Vassiliev invariants discussed.

1. INTRODUCTION

I describe recent results on canonical quantum gravity, knots, and spin network invariants developed in collaboration with J. Pullin from the Center for General Relativity and Gravitation of The Pennsylvania State University and J. Griego from Universidad de la República, Montevideo. More details can be found in refs. 1–3.

In recent years, the kinematics [4] of canonical loop quantum gravity, which describes the space or solutions of six of the seven Ashtekar constraints, has been well understood and has led to several appealing results. However, we do not have a fully satisfactory control of the dynamics and the Hamiltonian constraint. If a dynmical description is not found the kinematical setup should be abandoned, no matter how appealing are the physical results.

Let us recall the main results. Starting from the new variables introduced by Ashtekar, one can construct a representation of the quantum theory purely in terms of loops (the loop representation), and a suitable basis for describing quantum states is given by spin networks. That is, quantum states are labeled

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by spin networks, multivalent graphs embedded in three-dimensional space, with a system of weights associated to each side of the graph. If one imposes the diffeomorphism constraint, one considers functions of the diffeomorphism class of a spin network. There exists a precise sense in which one can endow such a space with an inner product [4], in terms of which spin network states are orthonormal.

In this context, there exists a proposal for the action of the Hamiltonian constraint of quantum gravity due to Thiemann [5]. In such a proposal the Hamiltonian constraint weighted by a lapse is just given by a topological operator acting at the vertices of the spin networks. The Hamiltonian constraint commutes with itself, as is expected should happen in a diffeomorphism-invariant context. It is not possible to check the constraint algebra in the non-diffeomorphism-invariant kinematical space of cylindrical functions because the diffeomorphism generator is not well defined there. However, it is possible to enlarge the diffeomorphism-invariant space [6] in such a way that Thiemann's Hamiltonian is still well defined. One can show that in this bigger space the Hamiltonian constraint continues to commute with itself. This suggests that an inconsistency could appear, since the right-hand side of the classical Poisson bracket vanishes only when the diffeomorphism constraint is satisfied. Even though one can arrange the right-hand side of the commutator to also vanish, it appears that the price to pay is tantamount to having a degenerate metric [7]. It also appears that this result is rather generic, holding for many possible detailed forms of the action of the Hamiltonian at vertices. Other nonlocal proposals also seem to suffer of difficulties with the constraint algebra [8].

On the other hand, over the last few years, a variety of formal results has been obtained on a different space of sates in which the loop derivative is well defined. In particular, a proposal for the Hamiltonian constraint of quantum gravity in terms of loop derivatives exists, and it has been shown that at a formal level the classical Poisson brackets are reproduced by the quantum theory [9]. The main drawback of this space of states is twofold: on one hand there is the fact that the loop derivative does not appear to exist on generic diffeomorphism-invariant states. This is due to the fact that such states change discontinuously when one changes the loops and therefore one cannot introduce a differential operator in loop space. Moreover, definitions of the Hamiltonian in terms of the loop derivative have only been attempted in the context of multiloops (not spin networks). In this context, each type of loop and intersection has to be treated individually, and therefore most results (for instance, proofs that certain states were annihilated by the constraints) were only of a partial nature.

The loop derivative [10] is a differential operator in loop space that arises by considering two loops that differ by an infinitesimal element of area as "close." It acts on basepointed objects (either loops or spin networks) by adding a path starting at the basepoint up to a point x, where it introduces an infinitesimal planar loop, and retraces back to the basepoint. The definition is

$$(1 + \sigma^{ab}\Delta_{ab}(\pi_o^x))\Psi(\gamma) = \Psi(\gamma \circ \delta\gamma) \tag{1}$$

where π is a path from the origin to x, and the loop $\gamma \circ \delta\gamma$ is shown in Fig. 1. Here $\sigma^{ab} \doteq \delta u^{[a}\delta v^{b]}$ is the infinitesimal element of area spanned by the two vectors u and v defining the parallelogram one adds at the end of the path π. For a spin network acting at a line on the network, the definition is exactly the same as for loops, except that the path becomes a line of the same valence as the line on which the basepoint lies, and the infinitesimal loop is also of the same valence as the path. It is important to notice that the loop derivative does not change the connectivity of a spin net. For instance, the derivative of a theta-net is a theta-net.

The purpose of this paper is to note that there exists a set of especially important knot invariants that appear to have the property of being loop differentiable. As a consequence, one can operate on them in terms of the Hamiltonian and diffeomorphism constraints of quantum gravity written in terms of loop derivatives in a more systematic way. The invariants in question are the Vassiliev invariants, which in turn are conjected [11] to be complete enough to be able to separate knots. Moreover, they have recently been generalized to the spin network context [12]. We will show explicitly that these invariants are annihilated by the diffeomorphism constraint of quantum gravity written in terms of loop derivatives. We will also start the analysis of the action of the regularized Hamiltonian constraint of the theory. All of this will be done in terms of spin networks, which allows us to discuss in a unified way all types of intersections and loops.

More specifically, in Section 2 we show that it is possible to define a denumerable set of ambient isotopy spin-knot invariants. These invariants are the natural extension to spin networks of the Vassiliev invariants. In Section 3 we show that this set is loop differentiable in the sense of distributions. In Section 4 we analyze the diffeomorphism constraint. Contrary to what happens in the space of cylindrical functions, the diffeomorphism constraint is well defined on an arbitrary loop-differentiable spin network. All the Vassiliev s-knot invariants solve this constraint. Finally, in Section 5 the regularized Ashtekar Hamiltonian constraint is defined in the space of loop-differentiable

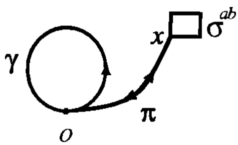


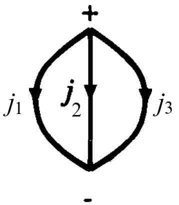
Fig. 1. The loop that defines the loop derivative.

spin networks. We show that the Chern–Simons state in the spin network representation solves the constraint with cosmological constant, and we compute the explicit action of this constraint on the Vassiliev invariants.

2. PRELIMINARIES

2.1. Spin Networks

Spin networks are constructed considering graphs that are embedded in three dimensions. The graphs are multiconnected with intersections that can be trivalent or of higher valence. Each connecing line is associated with a holonomy of the connection A_a^i in a given representation of the group in question [in our case, $SU(2)$, representations are labeled by a (half)integer]. One can construct a generalization of the trace of the holonomy (Wilson loop) which we call a Wilson net, by considering the traces of the holonomies along the Wilson nets joined by appropriate "intertwiners" at the intersections. The intertwiners consist of invariant tensors in the group. An example of a Wilson net for the "theta" network is given by



$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \qquad U_{m_1}^{(j_1)n_1} U_{m_2}^{(j_2)n_2} U_{m_3}^{(j_3)n_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2}$$

There are several possibilities to connect the invariant tensors and the holonomies and this has led to different conventions in the definition of the spin networks. The convention we will choose will follows closely that of Witten and Martin [13, 14] and differs from those of other authors in the field [15, 5, 16]. For a detailed discussion see refs. 2 and 3. With the above definition, the Wilson net has certain properties. In particular there is a dependence of the Wilson net on the orientation of the vertices. We now modify the definition in order to have an invariant that does not depend on the orientation of the vertices. This will correspond to the normalization of the Wilson net given by Witten and Martin. What we do is multiply the definition introduced up to now times a factor given by

$$V_{\pm} = \exp\left(\pm \frac{i\pi}{2}[j_1 + j_2 + j_3]\right) \sqrt[4]{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)} \tag{3}$$

for each vertex in the spin network. With this factor, one can show that the Wilson net defined is invariant under changes from +- to −-type vertices.

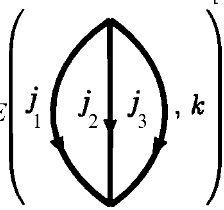
It is worthwhile noticing that in the work of Kauffman and Lins [16] on invariants of spin networks the normalization is different.

2.2. Knot Invariants for Spin Networks

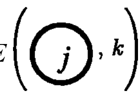
We wish to study invariants that represent the transform of the Chern–Simons state into the spin network basis. That is, we are interested in expressions of the form

$$E(\Gamma, k) = \int DA \exp(ikS_{CS}) W(\Gamma, A) \tag{4}$$

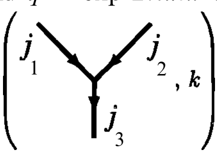
where Γ is a spin network and $W(\Gamma, A)$ is the Wilson net we introduced in the previous subsection. This kind of integral has been analyzed using monodromy techniques [13, 14] and variational techniques [1, 17]. The result is a regular isotopic invariant of spin networks. The techniques do not give a unique answer for the invariant, but there are several possibilities, tantamount to having several definitions for the measure DA. Each possibility is uniquely characterized by prescribing a value for the invariant on the so-called theta-net. The choice of Witten and Martin [13, 14] is

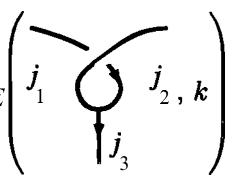
$\Theta(j_1, j_2, j_3) = E$  $= \sqrt{\Delta_1 \Delta_2 \Delta_3}$ (5)

with

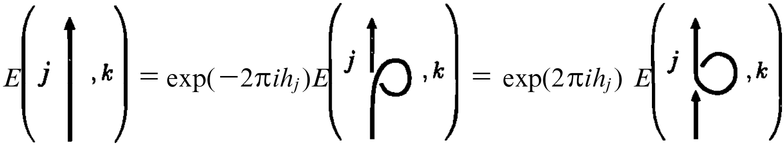
$\Delta_j = E$  $= \dfrac{q^{j+1/2} - q^{-j-1/2}}{q^{1/2} - q^{-1/2}}$ (6)

and $q = \exp 2\pi i/k$. We have

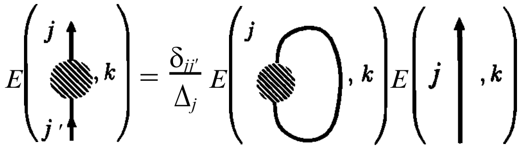
E 

$= (-1)^{j_1+j_2+j_3} \exp[i\pi(h_1 + h_2 - h_3)]\, E$  (7)

$$h_i \equiv \frac{j_i(j_i + 1)}{k} \tag{8}$$



(9)



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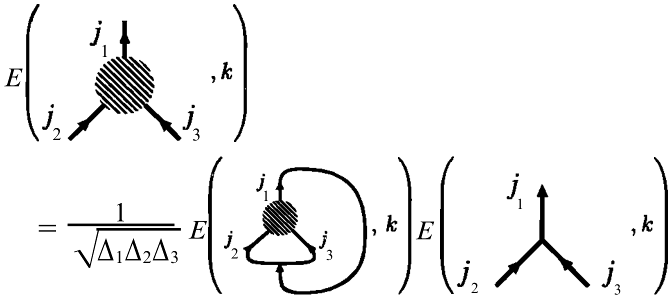

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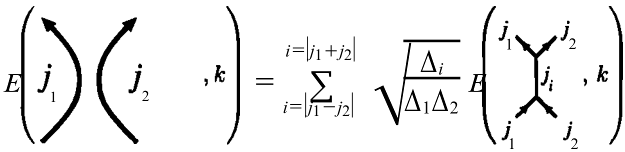

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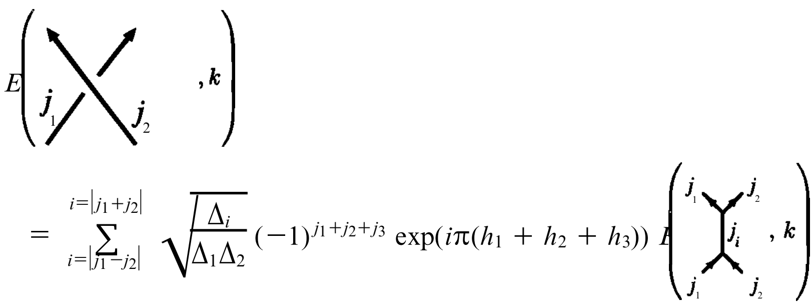
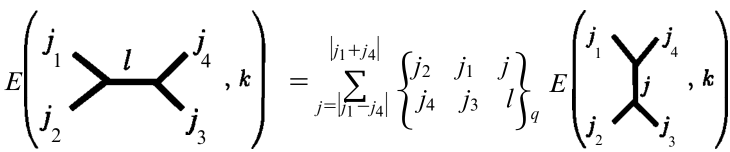
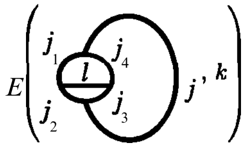


(13)



(14)

where the expression in curly braces is the q-deformed Racah symbol, which is defined as



$$\left\{\begin{matrix} j_2 & j_1 & j \\ j_4 & j_3 & l \end{matrix}\right\}_q = \frac{E(\cdot\,, k)}{\sqrt{\Delta_1\Delta_2\Delta_2\Delta_4}} \tag{15}$$

The above expressions completely characterize the invariant for any spin network. The explicit expression for the value of the invariant in the last expression, "the tetrahedron diagram," can be computed using the recoupling identity (14) in conjunction with the definitions of the delta and theta diagrams (6) and (5) (see ref. 14 for an explicit computation).

The moves (9) and (8) correspond to diffeomorphisms; therefore the invariant constructed is not invariant under diffeomorphisms. In other words, the invariant is a diffeomorphism invariant of ribbons and not of ordinary loops. One needs a framing (a prescription for assigning a ribbon to each loop) to have a well-defined invariant.

However, it was noted early on in the context of loops, and later in the context of spin networks [12], that the dependence on framing can be concentrated on an overall factor. In order to isolate this factor, one simply considers a power series expansion in $\kappa = (2\pi i)/k$ and extracts the linear coefficient in κ, $v_1(\Gamma)$. That coefficient, exponentiated, is the overall factor. This constructions is discussed in detail in ref. 12, where it is shown that it indeed leads to an ambient isotopic invariant, in other words, a genuine diffeomorphism-invariant function of loops. The resulting invariant is then given by

$$E(\Gamma, \kappa) = E(\Gamma, 0) \exp(v_1(\Gamma)\kappa)\, P(\Gamma, \kappa) \tag{16}$$

where $P(\Gamma, \kappa)$ is a spin network generalization of the Jones polynomial (it reduces to it when Γ is a simple loop in the fundamental representation) and v_1 is the first coefficient in the expansion in powers of κ (for a single loop it reduces to the self-linking number). The factor $E(\Gamma, 0)$, which corresponds to the evaluation of the invariant for $\kappa = 0$, contains information about the coloring of the graph, and no information about the embedding. It can be thought of as the evaluation of the Wilson net for a flat connection.

It was shown by Alvarez and Labastida [18] in the case of Γ being simple loops, and later generalized for links, that the invariant $E(\Gamma, \kappa)$ can be written as the exponential of a linear combination of primitive Vassiliev invariants,

$$E(\Gamma, \kappa) = E(\Gamma, 0) \exp\left(\sum_{i=1}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(\Gamma) r_{ij}(G) \kappa^i \right) \tag{17}$$

where $\alpha_{ij}(\Gamma)$ are the primitive Vassiliev invariants, which are only dependent on the embedding of the loop (they are independent of the gauge group of the Chern–Simons theory). d_i is the number of primitive Vassiliev invariants of order i, G is the gauge group [for our case $SU(2)$], and r_{ij} are group-dependent coefficients. We will assume that a similar structure appears for the case of generic $SU(2)$ spin networks, specifically

$$E(\Gamma, \kappa) = E(\Gamma, 0) \exp\left(\sum_{i=1}^{\infty} v_i(\Gamma) \kappa^i \right) \tag{18}$$

That is, we assume that the invariant is still given by the exponentiation of a set of "Vassiliev" invariants for spin nets, but we are not decomposing the expression in terms of "primitive" invariants, which is a structure that at present is not known in the spin network context. The $v_i(\Gamma)$ are ambient isotopic for $i > 1$.

By constructing these invariants of spin networks, we have bypassed one of the main obstacles that working with loops and links had in this context: how to comply with the Mandelstam identities. The invariants of spin nets we have constructed automatically take care of this. An important missing element in the spin network context is the idea of how "generic" is the set of $v_i(\Gamma)$. In the context of loops it is conjectured that the primitive Vassiliev invariants are enough to distinguish all knots. In the case of spin networks we are not working with primitive invariants and therefore it is questionable how generic the basis of invariants one is considering is. This is important if one is making the case that these invariants are the "arena" where one is going to discuss quantum gravity. If a decomposition in terms of primitive invariants of the exponential were known, the techniques we will develop will still be applicable. However, since it is not known how to do this decomposition, we will work with the v_i. It is worthwhile pointing out that the techniques we will introduce later in this paper to operate with loop derivatives and diffeomorphism constraints are geometrical in nature and not group dependent. Since it is known that all Vassiliev invariants can be constructed from the Chern–Simons integral with arbitrary groups, and our technique is not group dependent, we therefore have a *de facto* method to operate on all Vassiliev invariants. For concreteness we will concentrate on the case of $SU(2)$.

3. LOOP DIFFERENTIABILITY OF THE VASSILIEV INVARIANTS

We now wish to apply the loop derivative to the invariants we introduced in the previous section. *A priori* one expects that such a quantity does not exist. In particular, for a generic knot invariant, the loop derivative indeed does not exist. This is due to the fact that knot invariants are discontinuous functions in the space of loops. Loop derivatives change the topology of loops (for instance, they can remove intersections [19]), and therefore the limit defining the derivative is ill defined. What we will show here is that due to the properties of the invariants of Chern–Simons under deformations of the loops given by the skein relations, one can introduce a reasonable definition of the loop derivative for these kinds of invariants. It is similar to try to define the derivative of a discontinuous functions by allowing the derivative to take value in the distributions. We will analyze the consistency of this result with the properties of the invariants. The strategy is as follows. The invariants are defined as a functional integral of a Wilson net with a weight function. The only dependence on the spin net is in the Wilson net, so we will assume that the loop derivative of the invariant is equal to the functional integral of the action of the loop derivative on the Wilson net, with the appropriate weight factor,

$$\Delta_{ab}(\pi_o^x)E(\Gamma, \kappa) \equiv \int DA \exp(ikS_{CS}[A])\, \Delta_{ab}(\pi_o^x)W(\Gamma, A) \tag{19}$$

Here we have assumed that the limit defining the loop derivative and the one defining the path integral are interchangeable.

The starting point of the calculation is the action of the loop derivative on a holonomy [in the j representation of $SU(2)$] associated to an edge e_A^B of the spin network, containing the basepoint o,

$$\Delta_{ab}(\pi_o^x)U^{(j)}(e_A^B)^m_n = [U^{(j)}(e_A^o)U^{(j)}(\pi_o^x)F_{ab}^{(j)}(x)U^{(j)}(\pi^{-1}{}_o^x)U^{(j)}(e_o^B)]^m_n \tag{20}$$

and the fundamental relation satisfied by the Chern–Simons state,

$$F_{ab}^{(j)}(x) \exp(ikS_{CS}[A]) = -\frac{4\pi i}{k}\epsilon_{abc}\frac{\delta}{\delta A_c^{(j)}(x)} \exp(ikS_{CS}[A]) \tag{21}$$

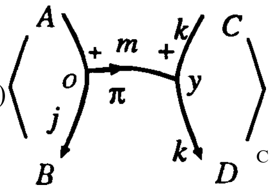
So the idea works exactly in the same way it did for ordinary loops [19, 1]; the loop derivative acting on a holonomy produces an F_{ab}, which can be reexpressed as a functional derivative acting on the exponential of the Chern–Simons form. The last step is to perform a formal integral by parts of the functional derivative and have it act on the holonomies of the spin network that were left by the action of the loop derivative. The end result is

$$\Delta_{ab}(\pi_o^x)E(\Gamma,\kappa)$$
$$= -2\kappa \sum_{e_C^D \in \Gamma} \epsilon_{abc} \int_{e_C^D} dy^c\, \delta^3(x-y)$$
$$\times \langle \cdots U^{(j)}(e_A^o)U^{(j)}(\pi_o^x)\tau^J_{(j)}U^{(j)}(\pi^{-1x}{}_o)U^{(j)}(e_o^B) \cdots U^{(k)}(e_C^y)\tau^J_{(k)}U^{(k)}(e_y^D)\cdots\rangle_{CS} \tag{22}$$

where $\tau^J_{()}$ are the $SU(2)$ generators[2] in the j representation, and the expectation value is assumed to be taken with respect to the measure $DA\exp(iS_{CS})$, and the dots refer to the fact that we just highlight the portion of the spin network where the loop derivative acts. It is understood that the products of holonomies continue until the net is closed and the appropriate traces are taken. A pictorial representation of the quantity within the expectation value is given in Fig. 2.

The above expression can be rearranged using the Fierz identity and the recoupling properties of $SU(2)$. One gets the final expression

$$\Delta_{ab}(\pi_o^x)E(\Gamma,\kappa)$$
$$= \sum_{e_C^D \in \Gamma} \sum_{m=0}^{2j} -\kappa\epsilon_{abc} \int_{e_C^D} dy^c\, \delta^3(x-y)\lambda_m^{\pm}(j,k)$$



(23)

where

$$\lambda_m^{\pm}(j,k)$$
$$= (-1)^{j\mp k} \sum_{l=|j-k|}^{j+k} (-1)^{l-(m\mp m)/2}(2m+1)(2l+1)\rho_l \begin{Bmatrix} j & j & m \\ k & k & l \end{Bmatrix} \tag{24}$$

So the end result is that the loop derivative acting on the invariant $E(\Gamma,\kappa)$ is nonvanishing only if the endpoint of the path π_o^x of the loop

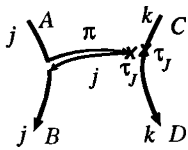


Fig. 2. The loop derivative acting on a spin network.

[2] Our convention for the $SU(2)$ generators is to take the Pauli matrices divided by 2. This differs from other authors [5].

derivative falls upon one of the lines of the spin network. The end result is, up to a factor, proportional to the invariant $E(\Gamma', \kappa)$, where Γ' is a new graph obtained by adding the path π_o^x to the original graph Γ. Notice that the action of the loop derivative is covariant under diffeomorphisms, in the sense that a diffeomorphism would shift both the graph Γ and the path π_o^x and therefore would act as a diffeomorphism on the graph Γ'. It is also worthwhile noticing that the general form of equation (23) is true for any gauge group; the only differences would appear in the recoupling coefficients λ and the corresponding irreducible representations associated with the graph. This emphasizes the geometrical nature of the action of the loop derivative and may open possibilities for generalizing this construction for invariants that do not necessarily arise from $SU(2)$ groups. The importance of this is that it appears that Vassiliev invariants for spin nets might arise as linear combinations of products of "primitive" invariants associated only with the topological embedding of the graph of the spin net, times some group-dependent factors that contain information about the valences of each line in the spin net. The loop derivative acts on such objects by ignoring the group prefactors and acting on the factors depending on the embedding of the spin net diagrams. In particular, if one acts on $E(\Gamma, 0)$, since it does not have information about the embedding, the loop derivative automatically gives zero.

The loop derivative as defined in this paper has several appealing properties that other differential operators in loop space do not have (see ref 10 for more details). One property we wish to emphasize is that the loop derivative satisfies Leibnitz' rule. It acts on a product of functions exactly like an ordinary derivative. This allows us, when evaluating operators on products of knots [as, for instance, when we extract the frame-dependent prefactor and the $E(\Gamma, 0)$], to perform explicit calculations.

For instance, we can compute the action of the loop derivative on any Vassiliev invariant of type v_n, by computing the logarithmic derivative of $E(\Gamma, \kappa)$. The final result is

$$\Delta_{ab}(\pi_o^x)v_n(\Gamma) = -\sum_{e_j \in \Gamma} \sum_{m=0}^{2j} \lambda_m \epsilon_{abc} \int_{e_j} dy^c \, \delta^3(x - y) \frac{E(\Gamma_m, 0)}{E(\Gamma, 0)} \frac{1}{(n-1)!} \times \left[\frac{d^{n-1}}{d\kappa^{n-1}} \exp\left(\sum_r (v_r(\Gamma_m) - v_r(\Gamma))\kappa^r \right) \right]_{\kappa=0}$$

and the action is nonvanishing only if o, x fall on two different lines of Γ, and the spin net Γ_m is obtained by taking Γ and adding to it the line π_o^x with valence m.

4. THE DIFFEOMORPHISM CONSTRAINT

In this section we will introduce the diffeomorphism constraint as a differential operator in loop space written in terms of the loop derivative. We will also show that the invariants we introduced in the previous section will be either annihilated by the constraint in the case of ambient isotopy invariants, or will transform appropriately in the case of regular isotopic invariants.

Let us consider the diffeomorphism constraint of quantum gravity written in terms of Ashtekar's new variables,

$$\hat{C}(\bar{\tilde{N}})\Psi[A] = \int d^3x\ N(x)^a \hat{\tilde{E}}_i^b(x) \hat{F}^i_{ab}(x)\Psi[A] \tag{26}$$

Because this expression involves the product of two operators at the same point, in general we need to regularize it, which we do via a point splitting function $\lim_{\epsilon\to 0} f_\epsilon(x, y) = \delta^3(x, y)$,

$$\hat{C}(\bar{\tilde{N}})\Psi[A]$$
$$= \lim_{\epsilon\to 0} \int d^3x \int d^3y\ f_\epsilon(x, y) N^a(x) \hat{\tilde{E}}_i^b(x) [U(\pi_x^y)\hat{F}_{ab}(y)U^{-1}(\pi_y^x)]^i \Psi[A] \tag{27}$$

where in order to preserve gauge invariance we have connected the F_{ab} and $\tilde{E}$ operators with holonomies along a path π going from x to y.

The above operator, when acting on a Wilson net, can be rewritten in terms of the loop derivative, as discussed in the context of loops [10]. The explicit expression is

$$\hat{C}(\bar{\tilde{N}})W(\Gamma, A)$$
$$= \lim_{\epsilon\to 0} \int d^3x\ f_\epsilon(x, y) \sum_{e_A^B\in\Gamma} \int_{e_A^B} dy^b\ N^a(y)\Delta_{ba}(\pi_y^x) W(\Gamma, A) \tag{28}$$

One can explicitly check that the action of this operator is a diffeomorphism, provided that the connection A is smooth. In the limit in which the regulator is removed, the path π shrinks to a point and one ends with a loop with an infinitesimal closed loop attached at the point x. The addition of this closed loop is tantamount to displacing infinitesimally the line of the original loop at the point x.

We will now assume that the generator of diffeomorphisms has in general the form given by equation (28) in terms of the loop derivative and we will show that when acting with it on the invariants from Chern–Simons we get the correct result. This result is nontrivial, since in the path integral that defines the invariant there are contributions from distributional connections.

To discuss in a cleaner fashion the action of the diffeomorphism operator on the invariants, we will consider diffeomorphisms in which the vector N has compact support. This will allow us to focus on the action of diffeomorphisms on individual edges, on vertices, etc., each at a single time. It is immediate that a generic situation can be analyzed combining all results we will derive. Let us start with the action at an individual line [we define $\hat{C}(\vec{N}) = \lim_{\epsilon\to 0} \hat{C}_\epsilon(\vec{N})$],

$$\hat{C}_\epsilon(\vec{N}) \left\langle \Big|\, j \right\rangle_{CS} = \kappa \sum_{m=0}^{2j} \lambda_{\bar{m}}(j, j) \int dy^b \int dz^c\, f_\epsilon(z, y) \epsilon_{abc} N^a(z)$$

$$\times \tfrac{1}{2}[\Theta(z - y) \left\langle \begin{smallmatrix} z- \\ y+ \end{smallmatrix} \Big|\!\!\supset m \right\rangle_{CS} {}_{j} + \Theta(y - z) \left\langle \begin{smallmatrix} y+ \\ z- \end{smallmatrix} \Big|\!\!\supset m \right\rangle_{CS} {}_{j}] \tag{29}$$

where Θ are Heaviside functions that order points along the line of the spin net we are considering. One can easily define them introducing a parametrization.

Now using recoupling and removing the regulator, assuming the following regularization function,

$$f_\epsilon(y, z) = \frac{3}{4\pi\epsilon^3} \Theta(\epsilon - |z - y|) \tag{30}$$

one gets

$$\hat{C}(\vec{N}) \left\langle \Big|\, j \right\rangle_{CS}$$

$$= \frac{\kappa}{8\pi} \sum_{m=0}^{2j} \lambda_{\bar{m}}(j, j) \frac{(-1)^{2j}}{(2j + 1)} \int_0^1 ds\, \epsilon_{zbc} \frac{\dot{N}(s)^a \dot{y}(s)^c \ddot{y}(s)^b}{|\dot{y}(s)|^3} \left\langle \Big|\, j \right\rangle_{CS} \tag{31}$$

where we have introduced a parametrization such that $dy^a = \dot{y}(s)^a\, ds$, and dots refer to total derivatives with respect to s. We can summarize the above result by saying that, for diffeomorphisms of compact support acting on lines of the spin net, we have that

$$\hat{C}(\overline{N})E(\Gamma, \kappa) = \kappa\mu(j)w(\overline{N})E(\Gamma, \kappa) \tag{32}$$

where $\mu(j) = \sum_{m=0}^{2j} \lambda_m^-(j, j)(-1)^{2j}/(2j + 1)$ is a recoupling factor, which depends on the weight of the line in question; $w(\overline{N})$ is the writhe introduced in the line by the action fo the diffeomorphism along the vector $\overline{N}$. We therefore see that the diffeomorphism has a well-defined limit. The result is dependent on a background metric through the writhe, as expected for a regular isotopic invariant. Notice that the result decomposes as a product of a factor depending entirely on the "coloring" of the spin net and another factor depending only on the embedding of the net in three dimensions. What we have therefore recovered is a formula that contains the information about the noninvariance of $E(\Gamma, k)$ under the addition of twists, given by the skein relation (9). This skein relation could be reobtained exactly by exponentiating the action of the loop derivative, as discussed in ref. 1, but we will not repeat the calculation here for brevity. The action of the diffeomorphism at an intersection gives a similar contribution.

Here we will show that the diffeomorphism operator we introduced vanishes when acting on the ambient isotopic invariants, concretely $P(\Gamma, \kappa)$. Therefore, it annihilates every Vassiliev invariant. In order to compute the action, let us recall equation (18),

$$E(\Gamma, \kappa) = E(\Gamma, 0)\left(1 + v_1(\Gamma)\kappa + (2v_2(\Gamma) + v_1(\Gamma)^2)\frac{\kappa^2}{2} + \cdots\right) \tag{33}$$

and analyze the action of the diffeomorphism on $E(\Gamma, \kappa)$ order by order in κ. To order zero we get,

$$\hat{C}(\overline{N})E(\Gamma, 0) = 0 \tag{34}$$

which is correct, since $E(\Gamma, 0)$ does not depend on the embedding of the spin net, just on its coloring, and it is immediately annihilated by the loop derivative. At the next order, we have

$$\hat{C}(\overline{N})v_1(\Gamma) = \mu(j)w(\overline{N}) \tag{35}$$

where we have assumed the diffeomorphism as acting on an isolated line of valence j; a similar formula holds for the intersections with a different coloring weight. As before, $w(\overline{N})$ is the writhe induced in the line by the vector field $\overline{N}$. If we now take advantage of the fact that the diffeomorphism operator satisfies Leibnitz' rule, we see that combining (32) and (35), we get

$$\hat{C}(\overline{N}) \exp(-\kappa v_1(\gamma))E(\Gamma, \kappa) = 0 \tag{36}$$

and therefore we see that indeed the diffeomorphism constraint annihilates all the su(2) Vassiliev invariants, namely,

$$\hat{C}(\overline{N})P(\Gamma,\kappa) = \hat{C}(\overline{N}) \exp\left(\sum_{n=2}^{\infty} v_n(\Gamma)\kappa^n\right) = 0 \tag{37}$$

The diffeomorphism constraint is defined on arbitrary Chern–Simons states, not only on the ones related to the $SU(2)$ group. This allows us to show that it annihilates all Vassiliev invariants and any function of the Vassiliev invariants. As we mentioned before, there are indications that Vassiliev invariants may constitute a basis of diffeomorphism-invariant functions of loops. The definition of the loop derivative we introduced in the spin network context is compatible with the fact, namely it naturally leads to a diffeomorphism constraint that annihilates explicitly all Vassiliev invariants.

5. HAMILTONIAN CONSTRAINT

Let us now consider the double-densitized Hamiltonian constraint of quantum gravity, possibly with a cosmological constant Λ,

$$\hat{H}(M) = \int d^3x \, Mg_{\epsilon,\epsilon',\epsilon''}(u, v, w, x)H(u, v, w)$$

$$\equiv \hat{H}(M) = \int d^3x \, Mg_{\epsilon,\epsilon',\epsilon''}(u, v, w, x)\epsilon^{ijk}\hat{\tilde{E}}_i^a(u)\hat{\tilde{E}}_j^b(v)[U(\pi_u^w)F_{ab}(w)U(\pi_w^u)]^k \tag{38}$$

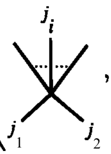
So we have chosen to join the F_{ab} with one of the E's via a holonomy. There are many other possibilities (for instance, joining all operators via holonomies, which would yield a gauge-invariant regularization) and they all yield the same classical expression in the limit $\epsilon \to 0$. A possible regulating function is defined as

$$g_{\epsilon,\epsilon',\epsilon''}(u, v, w, x) = f_\epsilon(x, u)f_\epsilon\left(x, \frac{u+v}{2}\right)f_\epsilon\left(x, \frac{u+v+w}{3}\right) \tag{39}$$

with f defined as in the diffeomorphism constraint. This regulating function is the same as the one usually considered for the "Ashtekar–Lewandowski" volume operator [4, 20].

To determine the action of the Hamiltonian in terms of spin networks, we consider the action of the operator we have just defined on a Wilson net. As is well known, this operator only acts at intersections of the net. At regular points it gives rise to "acceleration" terms, which we will omit since they vanish on diffeomorphism-invariant states [21, 10]. The nonvanishing action comes from the triads acting at two points on different strands i, k entering an intersection V,

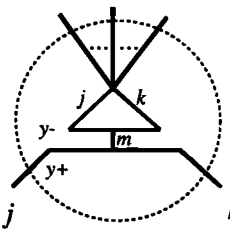
$$H(u, v, w)W\left(\right.$$



$$\left. , A \right)$$

$$= -\frac{i}{2} \sum_{i \neq k \in V} \sum_{m=|i-k|}^{i+k} (2m + 1)\rho_m(i, k) \int_{e_i} dy^a\, \delta^3(u - y) \int_{e_k} dz^b\, \delta^3(v - z)$$

$$\times\, [\Delta_{ab}(\pi^w_{y+}) - \Delta_{ab}(\pi^w_{y-})]W\left(\right.$$



$$\left. , A \right) \qquad (40)$$

where in the last diagram all the structure occurs "at a point" and points $y\pm$ are identified.

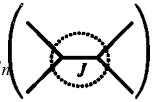
Having the Hamiltonian in terms of the loop derivative allows one to evaluate its action on any of the invariants we have been discussing. In particular one can show [3] that the Hamiltonian constraint with a cosmological constant term annihilates the invariant $E(\Gamma, \kappa)$.

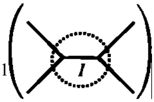
Another interesting result is that the Hamiltonian vanishes when acting on Vassiliev invariants at trivalent vertices. This can be straightforwardly seen by recalling the action of the loop derivative on a Vassiliev (25), and noting that at a triple intersection, using recoupling, $v_p(\Gamma_j) = v_p(\Gamma)$. We are allowed to use recoupling because the loop derivative acts by adding a line "at a point" when evaluated in the action of the Hamiltonian constraint. Therefore the Hamiltonian constraint vanishes at trivalent vertices.

Another important result concerning the action of the Hamiltonian on Vassiliev invariants can again be concluded from studying the explicit action of the loop derivative (25). Combining this expression with (40), one immediately concludes that the action of the "doubly densitized" Hamiltonian we are considering here is proportional to the "regularized volume" (strictly speaking, it is a volume squared) spanned by three of the tangents entering at an intersection [22],

$$\hat{\mathrm{Vol}}^{ijk}_{\mathrm{reg}}(x, \epsilon, \epsilon', \epsilon'') = \epsilon_{abc} \int_{e_i} dy^a \int_{e_j} dz^a \int_{e_k} dw^a\, g_{\epsilon,\epsilon',\epsilon''}(x, y, z, w) \quad (41)$$

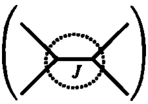
Concretely, acting on a four-valent vertex labeled by J, the action of the Hamiltonian is

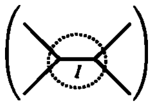
$$g_{\epsilon,\epsilon',\epsilon''}(x, u, v, w)\hat{H}_{\text{double}}(u, v, w)v_n$$


$$= \sum_I \sum_{i,j,k \in \text{Vertex}} \hat{\text{Vol}}^{ijk}_{\text{reg}}(x, \epsilon, \epsilon', \epsilon'')C^{IJ}V_{n-1} \tag{42}$$


with C^{IJ} is a proportionality factor depending on the valences of the strands entering the intersection. Similarly, the range of the sum in I is determined by the type of intersection. The invariant V_{n-1} is a nonprimitive Vassiliev invariant of order $n - 1$; it will involve sums of products of primitive Vassiliev invariants of lower orders.

This proportionality opens the attractive possibility of defining a "singly densitized" Hamiltonian by "dividing by the regularized volume." Having a singly densitized Hamiltonian can potentially lead to a much better defined operator, since its action could possibly be cast in a background-independent manner, as proportional to a Dirac delta. This possibility is currently under study. The action of the Hamiltonian could therefore be cast as

$$H_{\text{single}}(x)v_n$$


$$= \delta(x - \text{Vertex}) \sum_I d^{IJ}V_{n-1} \tag{43}$$


with d^{IJ} another constant factor. Having an operator with such a simple action is not only remarkable, but will open the possibility of further consistency checks, like computing the constraint algebra. All these issues are currently under study and evidently further work is needed to complete this program. The computation can be done "on shelf" by showing that the Hamiltonian commutes on Vassiliev invariants using the expressions introduced above. It could also be done "off shelf" since one has an expression for the Hamiltonian acting on any function that is loop differentiable, and therefore one can verify commutation relations with the diffeomorphism constraint.

6. CONCLUSIONS

We have defined an infinite set of spin network diffeomorphism invariants and given some hints about how to construct a complete set of them. They are given in terms of all the primitive Vassiliev invariants. We have shown that these invariants are loop differentiable, in the sense of distributions, and we have given a method to explicitly compute the loop derivatives. We have defined the Ashtekar constraints in the space of loop-differentiable functions and we have checked that the Vassiliev invariants are annihilated by the resulting diffeomorphism constraint. We have defined a double-densitized Hamiltonian constraint whose action on the space of Vassiliev invariants may be renormalized, but depends on the background metric used in the regulator. In order to find a fully consistent set of constraints, one needs to define a single-densitized Hamiltonian constraint and check the consistency of its algebra in the space of loop-differentiable functions.

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